An informational note on FINESSE: "Sidebands of Sidebands"

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Abstract

In FINESSE simulation, modulators generate sidebands only from the carrier, and not from other sidebands. Therefore, in some configurations (including, but not limited to, frequency-noise detection using Pound-Drever-Hall technique for a simple Fabry-Perot cavity), there can be a large error in the output depending on the optical parameters. This kind of error is not so frequently seen when you are simulating a typical interferometer configuration, but it's not completely negligible either. Why and when this happens are described. Some ideas for possible workarounds are also discussed.

1 Introduction

FINESSE is an excellent interferometer simulator developed by Andreas Freise[1], based on (also excellent) linear system simulator LISO developed by Gerhard Heinzel[2]. Its ability to allow one to configure almost any kind of interferometry configuration via simple file interface makes it an indispensable tool for any experimentalist (as well as for theorist, in my opinion) who is working on interferometric gravitational wave detector. It is for example really handy for experimenting several different parameters for a specific interferometer configuration, and it's also useful when you play around with your new idea like "well, let's place a mirror here and add another RF modulation there"[1].

However, at the moment FINESSE is not suitable for some kind of calculation including, but not limited to, frequency-noise detection using Pound-Drever-Hall technique for a simple Fabry-Perot cavity. What this "some kind" exactly is, as well as why it is so, are explained in the next section. Then a simple example is presented to show how large such an error can be. Finally some possible workarounds are discussed.

One special note: This must not be considered as any kind of bug in FINESSE. Rather, it's more a matter of knowing what we're doing.

2 When "sidebands of sidebands" are important

Sometimes it's useful to transmit some sidebands (with a frequency shift of, say, $\pm f_1$) through a phase modulator or EOM (which is driven at, say, f_2) and later measure the product of these transmitted sidebands and the sidebands generated by the EOM. One obvious example is the detection of frequency

[1] Such kind of "new" ideas seem to have some tendency to fail, but that's completely another story.

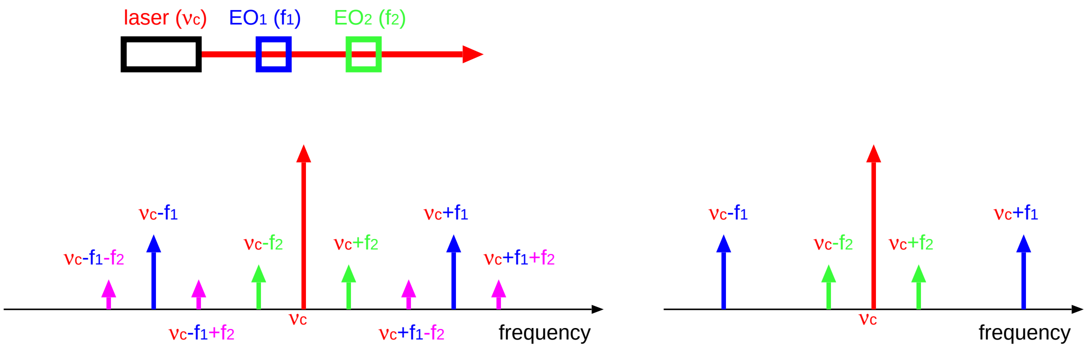

Figure 1: A laser and two EOMs in series in a real world (left) and in FINESSE world (right). In the latter, the second EOM only affects the carrier (ν_{C}), and the sidebands generated by the first EOM are not affected at all.

noise (or phase noise) of the laser using Pound-Drever-Hall technique. In this case, the "signal sidebands" around the carrier are, after passing an EOM and being reflected by a Fabry-Perot cavity, mixed with the RF sidebands created by the EOM. Under such circumstance (in a "real world" so to speak), the resulting signal is actually a sum of two distinct terms: One is the product of the sidebands that are generated from the carrier (i.e. $\pm f_1$ and $\pm f_2$), and the other is the product of the carrier (zero frequency shift) and the sidebands created from other sidebands ($\pm f_1 \pm f_2$).

As is described in FINESSE 0.82 DRAFT [1], EOM in FINESSE generates sidebands only for the carrier (I call this kind of sidebands "first generation" sidebands for convenience)[2], and the sidebands generated from other sidebands ("second generation" sidebands or "sidebands of sidebands") are simply "not created"[3]. Here are two equations (one for a real world and the other for a FINESSE world) for an optical field generated by a laser and two successive EOMs.

$$E_{\mathrm{real}} \simeq E_0 e^{i\Omega_c t} \left(1 + im_1 \cos\omega_1 t + im_2 \cos\omega_2 t - m_1 m_2 \cos\omega_1 t \cos\omega_2 t\right) \tag{1}$$

$$E_{\mathrm{FIN}} \simeq E_0 e^{i\Omega_c t} \left(1 + im_1 \cos\omega_1 t + im_2 \cos\omega_2 t\right) \tag{2}$$

where E_0, ω_c, m_i and ω_i $(i = 1, 2)$ are the amplitude and the angular frequency of the laser emission, the modulation indices and the modulation angular frequencies, respectively. Figure 1 graphically shows the difference.

By taking the square of the absolute value of the above two expressions, we see that there's no $\pm\omega_1 \pm \omega_2$ term in the intensity of the field in the "real" world thanks to the last term in Eq. 1, i.e. the second generation sidebands. This is because the product of the carrier and the second generation sidebands exactly cancels out with the product of the first generation sidebands (Fig. 2). This is not the case any more with FINESSE, as there's no "sidebands of sidebands" and therefore no cancellation (i.e. some intensity modulation is generated). You can easily confirm this by the following simulation:

```
# First example
```

[2] I first thought of calling it "first order", but it is too confusing as the word "order" in modulation/demodulation is so tightly connected to the orders of Bessel functions.

[3] This is quite reasonable when you are NOT going to observe the product of the first-generation sidebands generated by EOM and any kind of sideband components that are passing through this EOM (i.e. cross terms of $\pm f_1$ sidebands and $\pm f_2$).

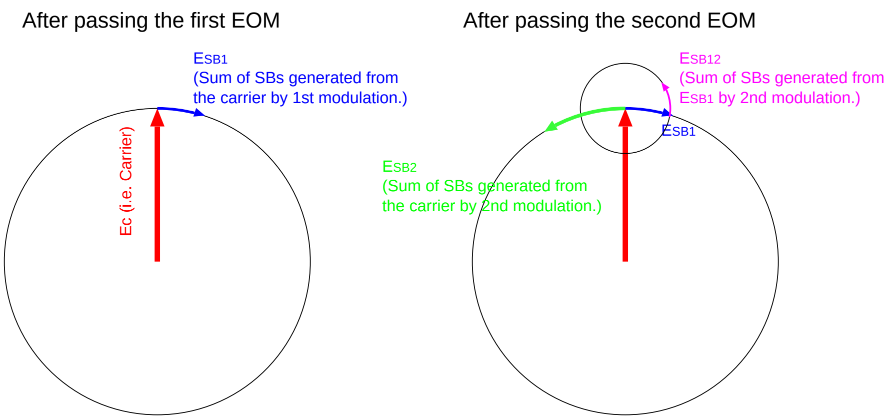

Figure 2: Phase diagram of the two successive modulation, in a real world. The cross terms of the two modulations are never found in the intensity because the product of the carrier (red) and the "sidebands of sidebands" (pink) cancels with the product of the first sideband (blue) and the second (green). The sum of the four (red, blue, green and pink) vectors in this figure still has the same amplitude with the original vector (red). In FINESSE, since the "sidebands of sidebands" (pink) are not there, an intensity modulation is automatically generated.

```
# You can find this at
# /afs/ipp/home/k/kwk/2002/0528/first.kat
l i1 1 0 n0 # laser P=1W, f_offset=0Hz
mod EO 10M 0.01 1 pm 0 n0 n1     # EO, 10MHz
mod EO2 60M 0.01 1 pm 0 n1 n2    # EO, 50MHz
pd1 pd2 70M 00  n2               # cross-term detection

xaxis EO phase lin 0 180 400     # sweep the phase of the first EO
yaxis abs                        # plot gain linearly
gnuterm x11                      # gnuplot terminal: x11
```

Usually we're interested only in first-order terms of modulation indices, therefore it may seem as if this is completely negligible as the resulting intensity modulation is $O(m_1m_2)$. However, if this $O(m_1m_2)$ is what we are looking at (e.g. if one of the sidebands is actually the signal sideband, if we are to do multiple demodulation for some weird configuration etc.), we have to be careful.

From what we've just seen, an error condition is summarized as this:

> When you transmit first-generation sidebands (either signal sidebands or RF sidebands in FINESSE's sense) through EOM, and later try to measure the cross terms of these transmitted sidebands and the first-generation sidebands created by the EOM, there's a possibility of a large error.

2.1 Signal

By the way, there's another modulation mechanism in FINESSE. By using `fsig` component, you can "shake" various kind of components in the interferometer. For example, by feeding signal to a mirror, you can actually generate phase modulation to the reflection as you "shake" the position of the mirror. An important aspect of `fsig` is that it generates the sidebands to each and every modulation sideband as well as the carrier in the field (but not to other signal sidebands). This means that the modulation mechanism for `fsig` is in some aspect closer to the reality (e.g. Eq. 1 than Eq. 2)[4]. Therefore the following conclusion is drawn:

> It is quite safe to make `fsig` act on any kind of modulation sidebands.

Up to here, everything is obvious and some may find this document boring. Such readers don't have to proceed further, as they know what they are doing. Others (if any), let's continue to see some more concrete example.

3 Example: PDH detection of the frequency noise

A simple example is presented here: Detection of frequency noise of the laser by using a Fabry-Perot cavity and Pound-Drever-Hall technique.

3.1 Basics

Before doing something by FINESSE, let's see a basic theory. Suppose that the frequency noise is sinusoidal with the angular frequency of ω_f. The laser field in this case is represented by an equivalent phase modulation like this:

$$E_\mathrm{i}(t) \sim E_0 e^{i\Omega t}\left[1 + i\frac{a_\mathrm{f}}{\omega_\mathrm{f}}\cos\omega_\mathrm{f} t\right] \tag{3}$$

where a_f is the "amplitude" of the angular frequency noise. We add an RF phase modulation using EOM:

$$E_\mathrm{EO}(t) \sim E_0 e^{i\Omega t}\left[1 + i\frac{a_\mathrm{f}}{\omega_\mathrm{f}}\cos\omega_\mathrm{f} t + im\cos\omega_\mathrm{m} t - m\frac{a_\mathrm{f}}{\omega_\mathrm{f}}\cos\omega_\mathrm{m} t\cos\omega_\mathrm{f} t\right] \tag{4}$$

where m is the modulation index and ω_m is the angular frequency of modulation.

We inject this field into a Fabry-Perot cavity which is exactly on resonance with the carrier. We represent the frequency-dependent complex reflectance of this cavity by

$$r_\mathrm{c}(\omega) \tag{5}$$

where ω is the difference of the angular frequency of the carrier and the field component in question. For carrier, this reflectance is

$$r_\mathrm{c}(0) \equiv r_0. \tag{6}$$

[4]However, `fsig` cannot generate higher-order sidebands, while modulator can, therefore in this aspect modulators are more realistic. We have to learn when to use which.

For convenience, we assume that the sidebands generated by the EOM are completely off-resonant with the cavity, and the same is true for the product of the FM noise sidebands and the RF sidebands:

$$r_{\mathrm{c}}(\pm\omega_{\mathrm{m}}) \sim r_{\mathrm{c}}(\pm\omega_{\mathrm{m}} \pm \omega_{\mathrm{f}}) \equiv r_{\mathrm{anti}}. \tag{7}$$

Also, assuming that the frequency of the FM noise is much smaller than the FSR of the cavity, the reflectance for the noise sidebands are written by

$$\begin{aligned} r_{\mathrm{c}}(\pm\omega_{\mathrm{f}}) &\sim r_0 - \frac{r_{\mathrm{e}} T_{\mathrm{f}} F}{\pi} \frac{\pm i\omega_{\mathrm{f}}/\omega_{\mathrm{c}}}{1 \pm i\omega_{\mathrm{f}}/\omega_{\mathrm{c}}} && (8)\\ &\equiv r_0 + \delta r(\pm\omega_{\mathrm{f}}) && (9) \end{aligned}$$

where r_{e}, r_{f}, T_{f}, F and ω_{c} are the amplitude reflectance of the end mirror, the amplitude reflectance of the front mirror, and the power transmittance of the front mirror, the finesse, and the cut-off angular frequency of the cavity[5].

By using Eqs. 4, 6, 7 and 9, the reflected field is represented by

$$\begin{aligned} E_{\mathrm{ref}} = E_0 e^{i\Omega t} \Big\{ & r_0 - r_{\mathrm{anti}} m \frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos\omega_{\mathrm{m}} t \cos\omega_{\mathrm{f}} t \\ & + i r_0 \frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos\omega_{\mathrm{f}} t + i\,|\delta r(\omega_{\mathrm{f}})|\,\frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos[\omega_{\mathrm{f}} t + \phi(\omega_{\mathrm{f}})] + i r_{\mathrm{anti}} m \cos\omega_{\mathrm{m}} t \Big\} \end{aligned} \tag{10}$$

where $\phi(\omega_{\mathrm{f}})$ is defined by using the imaginary ($\Im$) and the real ($\Re$) part of δr as

$$\tan\phi(\omega_{\mathrm{f}}) \equiv \frac{\Im[\delta r(\omega_{\mathrm{f}})]}{\Re[\delta r(\omega_{\mathrm{f}})]}. \tag{11}$$

We are interested in the terms with the angular frequency of $\pm\omega_{\mathrm{f}} \pm \omega_{\mathrm{m}}$:

$$\begin{aligned} I_{\mathrm{cross}} &\equiv -2\,|E_0|^2\, r_0 r_{\mathrm{anti}} m \frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos\omega_{\mathrm{m}} t \cos\omega_{\mathrm{f}} t \\ &\quad + 2\,|E_0|^2\, r_{\mathrm{anti}} m \cos\omega_{\mathrm{m}} t \left[r_0 \frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos\omega_{\mathrm{f}} t + |\delta r(\omega_{\mathrm{f}})|\,\frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos[\omega_{\mathrm{f}} t + \phi(\omega_{\mathrm{f}})] \right] && (12)\\ &= +2\,|E_0|^2\, r_{\mathrm{anti}} m\, |\delta r(\omega_{\mathrm{f}})|\,\frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos\omega_{\mathrm{m}} t \cos[\omega_{\mathrm{f}} t + \phi(\omega_{\mathrm{f}})]. && (13) \end{aligned}$$

This is the expression of a real-world signal we'd measure by using photodiode. For example, by measuring this intensity by a photodiode and demodulating at the same frequency as the RF phase modulation ($f_{\mathrm{m}} = \omega_{\mathrm{m}}/2\pi$), we get the "error signal" which is proportional to

$$|E_0|^2\, r_{\mathrm{anti}} m\, |\delta r(\omega_{\mathrm{f}})|\,\frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}} \cos[\omega_{\mathrm{f}} t + \phi(\omega_{\mathrm{f}})]. \tag{14}$$

The transfer function from the frequency noise to the error signal is the ratio of the above equation to the signal fed to the laser, i.e. $a_{\mathrm{s}} \cos\omega_{\mathrm{f}} t$. Since the amplitude of modulation a_{f} is a number with a dimension of angular frequency, the resulting transfer function has a dimension of ω^-1 (times laser power). In a complex plane this is:

$$H(\omega_{\mathrm{f}}) = -i\,|E_0|^2\, r_{\mathrm{anti}} m \frac{r_{\mathrm{e}} T_{\mathrm{f}} F}{\pi\omega_{\mathrm{c}}} \frac{1}{1 + i\omega_{\mathrm{f}}/\omega_{\mathrm{c}}}. \tag{15}$$

[5] Here I'm using the notational convention where the reflectance of a simple mirror is a real number (thus having the different sign for different surfaces). However, the use of this convention has nothing to do with the result of the calculation. If you prefer you can say the reflectance of a simple mirror is imaginary (thus the same sign for different surfaces), and you still get the same results.

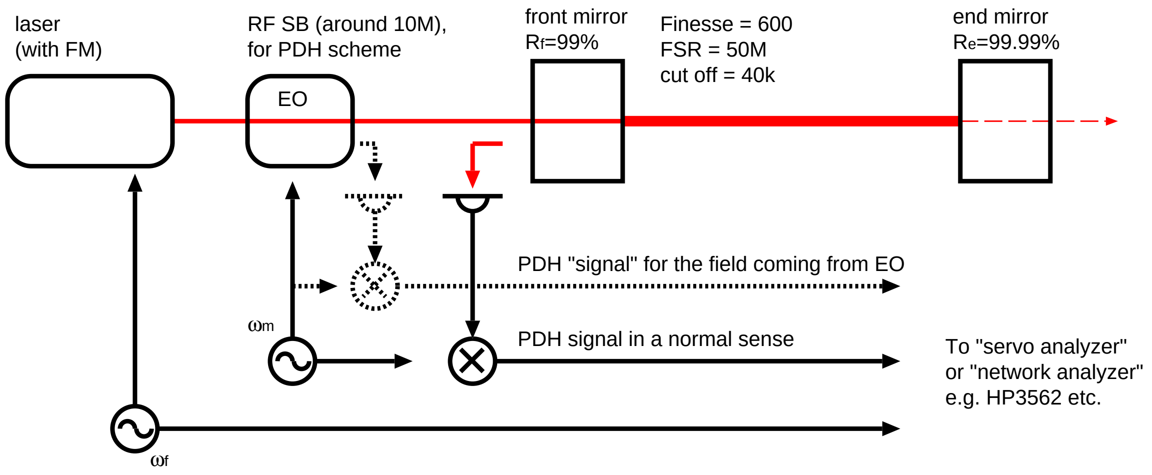

Figure 3: Apparatus of FINESSE simulation.

3.2 FINESSE simulation

As easily understood from what we have already seen before, it is possible to do the wrong simulation if you are not careful enough. This is because the second term in Eq. 10 doesn't exist in FINESSE world, and thus the resulting intensity would become

$$I_{\mathrm{cross}}^{\mathrm{FIN}} \equiv 2\left|E_0\right|^2 r_{\mathrm{anti}} m \frac{a_{\mathrm{f}}}{\omega_{\mathrm{f}}}\left\{\left|\delta r(\omega_{\mathrm{f}})\right| \cos\left[\omega_{\mathrm{f}} t+\phi(\omega_{\mathrm{f}})\right]+r_0 \cos \omega_{\mathrm{f}} t\right\} \cos \omega_{\mathrm{m}} t. \tag{16}$$

The first term in the curly brackets is the "real" signal, and the second is the pseudo signal generated by the lack of "sidebands of sidebands" when using EOM. In this specific setting of PDH technique, the error term would be negligible if, and only if, the following condition is met:

$$|r_0| \ll |\delta r(\omega_{\mathrm{f}})| = \frac{r_{\mathrm{e}} T_{\mathrm{f}} F}{\pi} \frac{\omega_{\mathrm{f}}/\omega_{\mathrm{c}}}{\sqrt{1+(\omega_{\mathrm{f}}/\omega_{\mathrm{c}})^2}} \tag{17}$$

Therefore, for example, when I want to measure the response of the almost-reflective Fabry-Perot cavity to the frequency noise, I'm likely to make a mistake.

Let's see such an example. The apparatus is shown in Fig. 3. There's a laser with frequency modulation, an EOM, two mirrors and two detectors. One detector is used to measure the Pound-Drever-Hall error signal in a normal sense (i.e. the signal obtained from the field coming from the cavity), and the other is to measure the pseudo signal obtained from the field coming from the EOM.

3.2.1 FINESSE source

The following is the straightforward implementation of the model we have just described: two mirrors, a laser with the frequency noise, an EOM and a photodetector. FM noise is implemented by the `fsig` component, and the signal sidebands are transmitted through EOM.

```
#----------------------------------------------------------------------------
# bad example
```

```
# This file is found at
# /afs/ipp/home/k/kwk/2002/0528/bad.kat
#
#                        m1                                m2
#         .-----.       .-.                               .-.
#         |     |       | |      . . . . . . . . . .      | |
# --> n0  | EO  |  n1   | |  n2  .      s1          .  n3  | |  dump
#         |     |       | |      . . . . . . . . . .      | |
#         `-----'       | |                               | |
#                       `-'                               `-'
#-----------------------------------------------------------------------------

# Definition of nearly all-reflective cavity.
m m1 0.99 0.01 0 n1 n2 # mirror R=0.99 T=0.01, phi=0
s s1 2.99792458 n2 n3 # space  L=c*10ns
m m2 0.9999 0.00001 0 n3 dump # mirror R=0.9999 T=0.00001 phi=0
# i.e. the finesse is 600 or something,
# FSR=50M,
# and the half BW is about 40k
# laser P=1W, f_offset=0Hz
l i1 1 0 n0
# FM for laser
fsig sig1 i1 10 0

# PDH modulation
mod EO 10M 0.01 1 pm n0 n1

# photo diode + 2 mixers for the field
# first demodulation at the exact frequency of EO, phase=0
# second is used for network analyzer.
pd2 reflected 10M 0 10 n1 # coming from m1:
pd2 incoming 10M 0 10 n1* # coming from EO:

xaxis sig1 f log 10 1M 400 # Change the FM frequency from
# 10 to 1M

xparam reflected f2 1 0 # Tune the photodetectors to follow EO
xparam incoming f2 1 0

yaxis db:deg  # plot gain in dB and phase
```

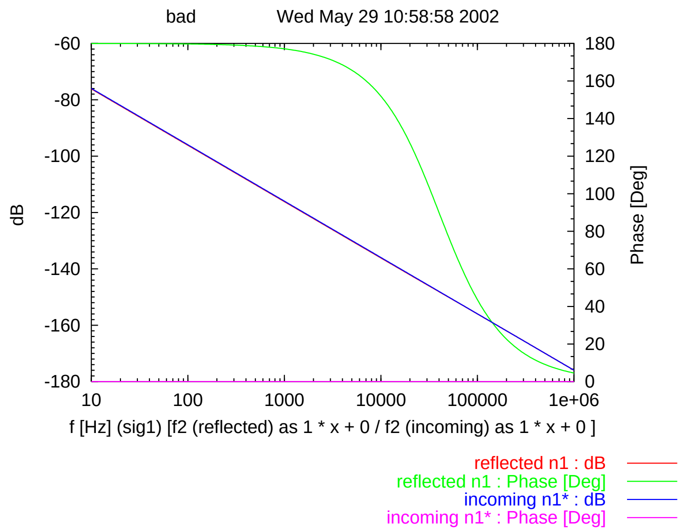

Figure 4: Transfer function from the frequency noise of the laser to the resulting Pound-Drever-Hall error signal for the reflected field ("reflected") using FINESSE. Also plotted is the transfer function from the frequency noise to the PDH signal for the field right after EOM ("incoming"). The absolute value of the amplitude for the two plots are almost identical. The "real" plot should be the low-pass like response with the DC level of -142dB [W · sec] and the corner frequency of 40kHz.

3.2.2 Result

The resulting plots are shown in Fig. 4. This seems to be odd if you don't know what's happening inside FINESSE, but, as you know it, it's actually clear. Since in this case the cavity is almost completely reflective, the resulting transfer function is dominated by the pseudo signal. More quantitatively, "almost reflective" leads to $|r_0| \sim 1$ and $r_{\rm e}T_{\rm f}F/\pi \sim 2$, therefore Eq. 17 is never met. From Eq. 15, the absolute value of "real" transfer function in this case should be approximately

$$\begin{aligned} |H| &\sim |E_0|^2 \frac{2m}{\omega_{\rm c}} \frac{1}{\sqrt{1+(\omega_{\rm f}/\omega_{\rm c})^2}} \\ &\sim 8\times 10^{-8}[\mathrm{W}\cdot\mathrm{sec}] \times \frac{1}{\sqrt{1+(\omega_{\rm f}/\omega_{\rm c})^2}}, \end{aligned} \tag{18}$$

The DC level should be at around -142dB.

3.3 Good example

Just to satisfy our curiosity, shown in Fig. 5 is the FINESSE plot for all-transmissive cavity with the identical mirrors (i.e. the reflectance as well as the transmittance of two mirrors are the same). In this case, since the DC reflectance of the cavity for the carrier is zero ($r_0 = 0$), Eq. 17 is satisfied and there's no pseudo signal. Being all-transmissive (i.e. $F \sim \pi/T_{\rm f}$), the transfer function is approximated

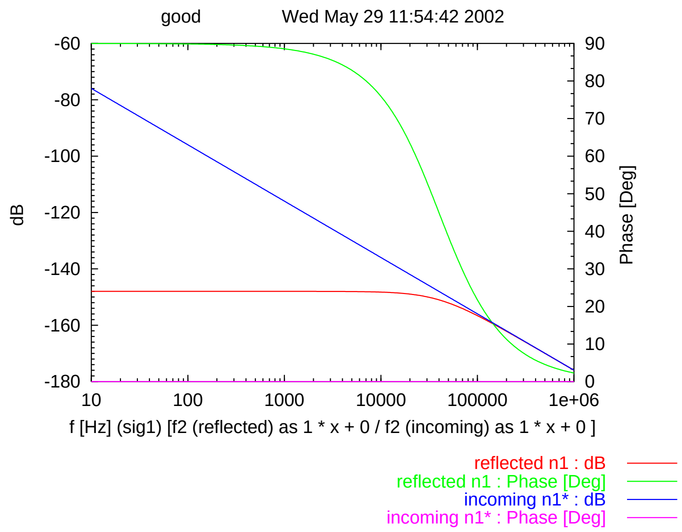

Figure 5: FINESSE plot for all-transmissive cavity. The only difference between this and Fig. 4 is that the mirrors of the cavity have the identical reflectivity in this case. In "reflected" signal, we cannot see the large pseudo signal this time. The plot shows a nice low-pass -filter-like response with the cutoff at around 40kHz, which is correct.

by

$$\begin{aligned} |H| &\sim \frac{m}{\omega_c}\frac{1}{\sqrt{1+(\omega_f/\omega_c)^2}} \\ &\sim 4\times 10^{-8}[\mathrm{sec}^{-1}]\times\frac{1}{\sqrt{1+(\omega_f/\omega_c)^2}}, \end{aligned} \tag{19}$$

where the DC level is -148dB [W $\cdot$ Sec], which agrees very well with the plot.

4 Workarounds

We've seen that in some cases it is possible that you get false results from FINESSE. Note that FINESSE has done nothing wrong, it's just us who used it in a wrong fashion. So what to watch out to avoid such kind of wrong usage?

4.1 In general, we'd be safer if we avoid these.

If possible at all, we'd better keep away from looking at the cross terms of the sidebands of different frequencies when (and only when) sideband of one frequency passes through the EOM that generates the other. From this, two variations come immediately, i.e. "avoid frequency modulation" and "avoid multiple RF demodulation".

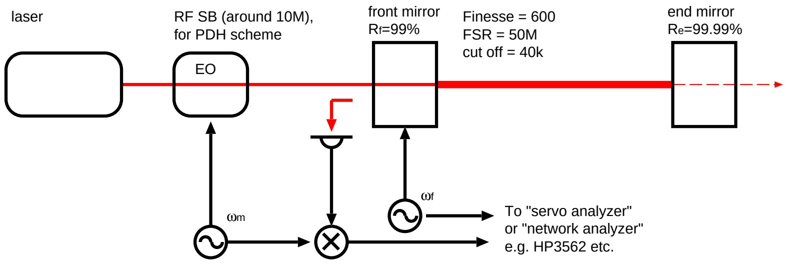

Figure 6: For such a simple experiment, I simply could have done this.

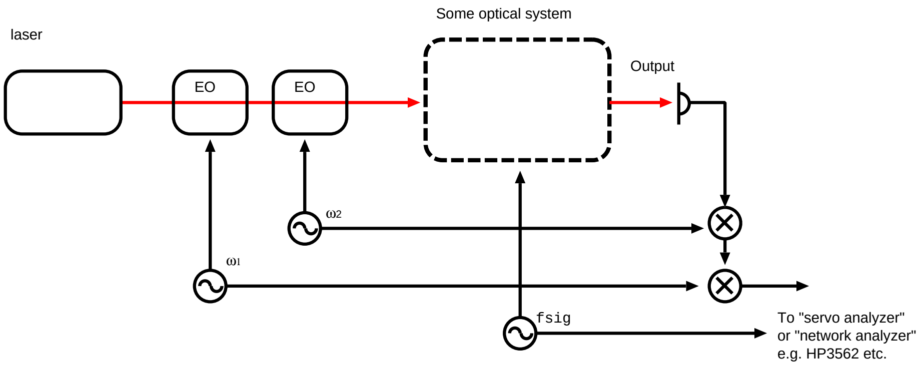

Figure 7: Avoid such kind of multiple RF demodulation if possible.

4.1.1 Avoid frequency modulation

Or, more precisely, avoid seeing something related to frequency noise. If you want to feed `fsig` to the laser, then you'd surely want to send it through at least one EOM, and later you'd like to see the product of signal sideband and the RF sideband. This is exactly where the problem can arise.

If possible at all, we'd better use some alternative. For example, for the example presented in this document, I could have shaken the mirror to see the cavity's response (Fig. 6).

4.1.2 Avoid multiple RF demodulation

What I'm saying here is "If there're two EOMs with the different frequencies (f_1 and f_2), it's safer to stay away from looking at the cross terms of these two that have the frequency of $\pm f_1 \pm f_2$ (Fig. 7).

4.2 If you cannot avoid anything

It might be that you cannot (or, more likely, don't want to) avoid these. In that case, several considerations will help.

4.2.1 Look at the signal demodulation phase

To look at your FINESSE plot critically, it's helpful to use your knowledge about the interferometry. If what we'd like to see is some sideband which is more or less resonant with some resonator and we

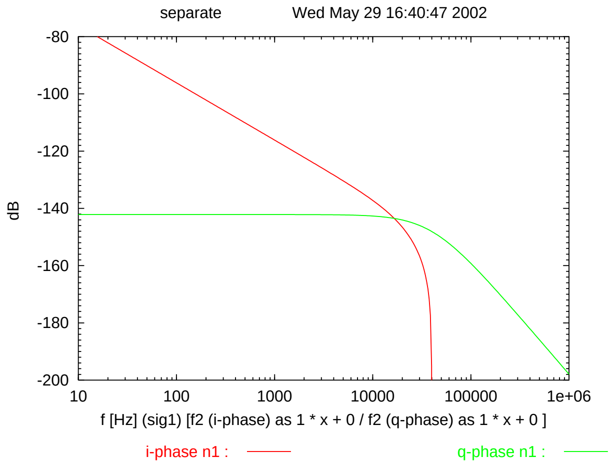

Figure 8: The same quantity as was shown in Fig. 4, but this time the plot is done separately for quadrature- and in-phase demodulation. Note that the term "quadrature" and "in" are used in relation to the signal, not the RF sidebands.

know for sure that "sidebands of sidebands" are off-resonant, or vice versa, then the pseudo signal appears in a different demodulation phase than the real signal (the term "demodulation phase" is used here in relation to the resonant fields, i.e. in this case the signal sidebands). It's impossible to prove this explicitly as the problem is too general, but in our example this is really easy to see.

Let's have a look at Eq. 9 again. The signal sidebands are "almost resonant", and this "almost, but not completely" status is represented by the term δr. At around the resonance, a resonator's response changes only in phase, meaning that δr is "orthogonal" to r_0 and r_{anti} in phase diagram (i.e. if r_0 is a real number then $\delta r(\omega_{\mathrm{f}})$ is imaginary for very small ω_{f}). By comparing Eq. 13 with Eq. 16, it turns out that the pseudo signal is proportional to two "parallel" quantities (r_0 and r_{anti}), while the real signal is proportional to two quantities (r_{anti} and δr) that are "orthogonal" to each other, which results in the difference in the demodulation phase ($\cos[\omega_{\mathrm{f}} t + \phi(\omega_{\mathrm{f}})]$ and $\cos \omega_{\mathrm{f}} t$).

It's a good idea to plot the in-phase and quadrature-phase demodulation separately on one graph. Figure 8 is the plot for the same signal as shown in Fig. 4 ("reflected"), but this time I plotted the in-phase and quadrature-phase demodulation separately instead of automatically generating the transfer function. The quadrature signal seems to be rather sensible while the in-phase signal is strange, indicating that something wrong is happening in the field component that is not usually taken into account.

4.2.2 Do some calculations

As we've seen, if your optical system is all-transmissive Fabry-Perot and what you'd like to see is the PDH signal caused by the frequency noise, then you don't have to care. There are of course many situations like that.

In order to judge if you're in trouble or not, probably it helps to do some additional calculations (with a help from FINESSE). The basic idea is to evaluate the amplitude of four parts in your field, i.e. the carrier (c), the first sidebands (SB1) generated from the carrier, the second sidebands (SB2) generated from the carrier, and the second-generation sidebands (SB12) generated from the first sidebands by the second modulation. For c, SB1 and SB2, you can get these values immediately by using FINESSE and the amplitude detector. For SB12, you can do it on paper as we have done for the simple example (but of course you can also use FINESSE to help your calculation). Finally, compare the product of c and SB12 with that of SB1 and SB2. If the former is non-negligible, you'll have some error in FINESSE calculation, therefore you have to do more work.

4.3 Tricky bits: Mimic sidebands of sidebands

If none of the above helps (i.e. you cannot avoid anything,cannot find any alternatives, and you are sure you'll have some error; in other words your case is an exceptional one), as a last resort it should be possible to mimic a real-world modulator to some extent. Though none of these techniques can be used for higher-order sidebands ("higher-order" as in higher-order Bessel function), sometimes these can be useful.

4.3.1 Using an EOM plus two AOMs

If you really want to use two modulators for multiple RF modulation/demodulation, it should be possible to substitute the second EOM by an EOM and two AOMs.

Let's return to Eq. 1 and Fig. 1. We know that, in FINESSE, all "sidebands of sidebands", namely the last term in Eq. 1 ("$\nu_c \pm f_1 \pm f_2$" vectors in the figure) is not there. If we rewrite this term as

$$-m_1 m_2 \cos\omega_1 t \cos\omega_2 t = -\frac{m_1 m_2}{2}[\cos(\omega_1+\omega_2)t + \cos(\omega_1-\omega_2)t], \qquad (20)$$

it is obvious that we can generate these sidebands separately from the carrier by using two additional modulators. Since these sidebands are always "parallel" to the carrier in the phase diagram (i.e. Eq. 20 is a real number), these modulations should be generated by the amplitude modulators (I say AOMs, as AOM is the most familiar amplitude modulator to me). All we have to do is to supply a set of precise modulation index, frequency and phase for each of additional AOMs, i.e. $m \sim m_1 m_2$, $\omega = \omega_1 \pm \omega_2$ and $\phi = 180$deg. This approximation for modulation index of AOM cannot hold true if either or both of the two modulation indices for EOMs are not much smaller than the unity; in that case you have to calculate the Bessel function by yourself.

Figure 9 shows the plot of the PDH detection of phase noise by using an EOM plus two AOMs[6]. In this case the "phase noise" was actually generated by another EOM. As can bee seen on the figure, there's no large error because "sidebands of sidebands" were correctly implemented by AOMs. However, we can still see that there's a small pseudo signal in the incoming field because of the incomplete cancellation, and actually the PDH signal is slightly affected by this error (see the phase plot in the lower frequency range). If this is a problem, we have to further fine-tune the modulation indices for AOMs.

4.3.2 Using an EOM plus four additional single sideband modulators

Modulators can be operated in "single sideband (SSB) mode" in FINESSE. Using four additional such modulators, we can also generate the sidebands of sidebands[7]. The difference in using four

[6] `/afs/ipp/home/k/kwk/2002/0528/aom.kat`

[7] This was suggested by Andreas Freise in response to my two-AOM technique.

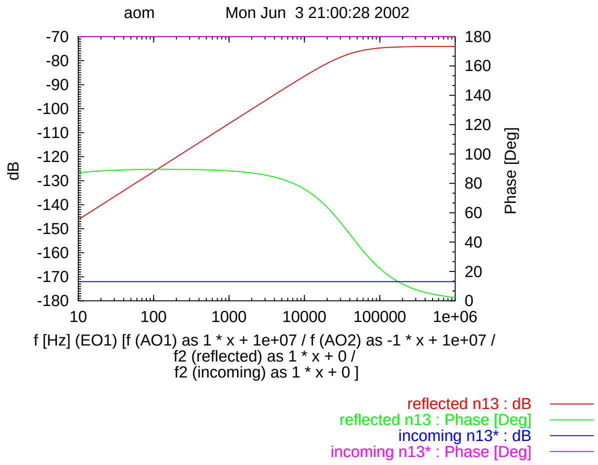

Figure 9: A plot obtained by using two additional AOMs. This time the "noise" is the sidebands generated by another EOM, so it is phase modulation rather than the frequency modulation (therefore the response is high-pass-filter-like). There's no large error because "sidebands of sidebands" are taken into account.

SSB modulators instead of two AOMs is that we can use "SSB EOM"s in the latter. In FINESSE, both EOM and AOM decrease the amplitude of the carrier that passes through themselves, but its dependence on the modulation index m is different: In EOM, the carrier amplitude is equal to the zero-th order Bessel function ($J_0(m) \sim 1 - m^2/4$), while in AOM it is a DC part plus linear term of the modulation index $(1 - m/2)$. In SSB mode, this changes to $[1 + J_0(m)]/2 \sim 1 - m^2/8$ for EOM and $1 - m/4$ for AOM, respectively [3]. On the other hand, in order to perfectly cancel the pseudo signal, any decrease in the carrier caused by the first modulator should be compensated by increasing the modulation index of the second modulator, which causes further decrease in carrier, and this "additional loss" should also be compensated by the third modulator etc. Since the additional loss in SSB EOM has a smaller dependence on modulation index than in AOM, using four SSB EOMs has a potential advantage of better pseudo signal cancellation over using AOMs, at least in theory.

Figure 10 shows the FINESSE simulation using four additional four SSB EOMs instead of two AOMs [8]. Apart from the modulators, there's no difference between this plot and the one shown in Fig. 9. Despite the potential merit of smaller loss, Fig. 10 shows the same order of error as Fig. 9.

This can be understood if we return to Eq. 2 to include the loss in the carrier:

$$\begin{aligned} E_{\mathrm{FIN}} =& E_0 e^{i\Omega_c t} [\ J_0(m_1)J_0(m_2) + 2iJ_1(m_1)\cos\omega_1 t + 2iJ_0(m_1)J_1(m_2)\cos\omega_2 t] \\ =& E_0 e^{i\Omega_c t} [\ J_0(m_1)J_0(m_2) + 2iJ_1(m_1)J_0(m_2)(1+\epsilon)\cos\omega_1 t + 2iJ_0(m_1)J_1(m_2)\cos\omega_2 t] \end{aligned} \tag{21}$$

Since the sidebands are not affected by the modulators in FINESSE, the second term in Eq. 21 is slightly different from its real-world counterpart, which is represented by the dimensionless error ϵ. If the second modulation index (m_2) is much smaller than the unity, ϵ is on the order of $O(m_2^2)$. Note that this error is irrelevant of the second-generation-sideband generation mechanism that we are

[8] `/afs/ipp/home/k/kwk/2002/0528/ssb.kat`

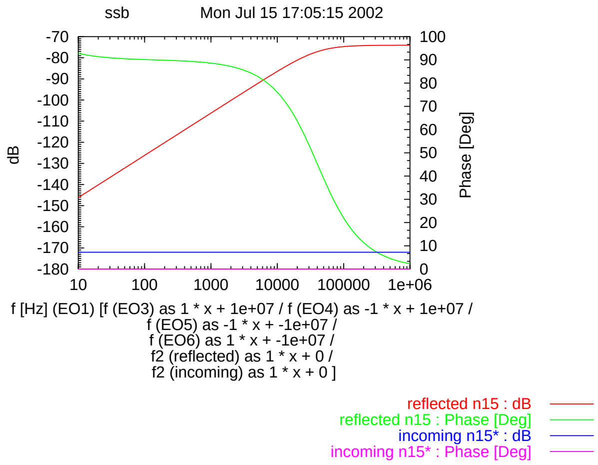

Figure 10: A FINESSE simulation using four additional SSB-EOMs instead of two AOMs. Apart from the modulation implementation, the setup is identical to that of Fig. 9. Despite the potential merit or smaller loss, this plot shows the same order of error as Fig. 9 (see the phase plot).

talking about. To achieve a perfect cancellation of the pseudo signal, actually this error should also be canceled by adjusting the amplitude of the second generation sidebands.

When we use some technique (either AOMs or SSB EOMs) to generate additional second-generation sidebands (without counting the error ϵ), that would introduce the error in the carrier as described before, the order of which is either $O(m_1 m_2)$ using AOMs or $O(m_1^2 m_2^2)$ using SSB-EOMs. This surely makes difference in the resulting error, but the net error is the sum of these errors and ϵ. Therefore, as far as m_1 and m_2 are on the same order, we cannot see any drastic difference between AOM and SSB-EOMs technique, as the additional AOM error is on the same order as ϵ.

4.3.3 Using a BS as an EOM

If you don't need to feed signal for any other purpose, you can use a beam splitter as an EOM. You have to connect `fsig` to a BS and use the reflection that contains the modulation. Since `fsig` acts on every modulation sideband, second generation sidebands are automatically taken care of.

The following is an example of FINESSE simulation on some variation of Telada's technique[4] (Fig. 11). RF sidebands close to the FSR of the cavity are used as "probes", and the reflection from the cavity is doubly demodulated. When the carrier is resonant with the cavity, the demodulated signal gives the deviation of the cavity's length from the probe RF wavelength.

```
#------------------------------------------------------------------------------
# Using BS as an EOM
# /afs/ipp/home/k/kwk/2002/0528/fsr3.kat
# "bad example" using two EOMs is fsr2.kat in the same directory.
# keita
#
#                                m1                       m2
```

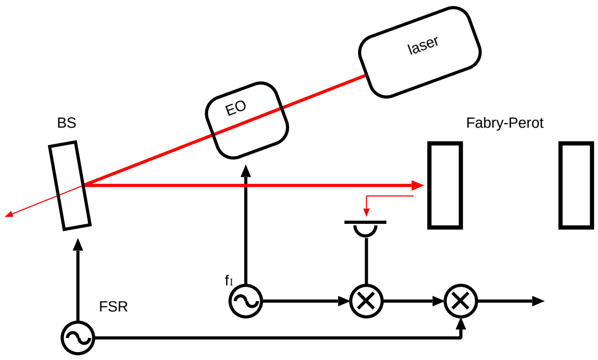

Figure 11: An example of the usage of beam splitter instead of EOM in FINESSE.

```
#                             .-.                      .-.
#                       refl. | |      . . . . .       | |
# --> n0 - EO - n11 - BS - n12 | |  n2  .  s1   .  n3  | |  dump
#                    / \      | |      . . . . .       | |
#                dump2  dump3 | |                      | |
#                             `-'                      `-'
#------------------------------------------------------------------------------

# Definition of nearly all-reflective cavity.
m m1 0.99 0.01 0 n12 n2 # mirror R=0.99 T=0.01, phi=0
s s1 2.99792458 n2 n3 # space  L=c*10ns
m m2 0.9999 0.00001 0 n3 dump # mirror R=0.9999 T=0.00001 phi=0
# i.e. the finesse is 600 or something,
# FSR=50M,
# and the half BW is about 40k

# laser P=1W, f_offset=0Hz
l i1 1 0 n0

# PDH modulation
mod EO 10M 0.01 1 pm n0 n11

# probe modulation
bs BS 0.99999 0.00001 0 0 n11 n12 dump2 dump3

#signal is connected to BS at 50MHz(!)
fsig signal BS 50M 0 0.005

# photo diode + 2 mixers for the field
pd2 inp 50M 0 10M 0 n12 # coming from m1:
```

```
pd2 quad 50M 90 10M 0 n12 # coming from m1:

xaxis signal f lin 49.9M 50.1M 400 # Change the probe frequency
xparam inp f1 1 0
xparam quad f1 1 0


yaxis abs  # plot gain in dB and phase

gnuterm x11
#gnuterm ceps  # gnuplot outputs to eps file
#------------------------------------------------------------------------------
```

Figure 12 shows the resulting plots, together with the ones generated by a straight-forward implementation (i.e.using two EOMs instead of one EOM and one BS). In "BS-modulation" plot we can clearly see that 50 MHz (i.e. FSR) is the null-signal frequency, while in "EOM-modulation"plot there's no such frequency.

5 Summary

Second generation sidebands, or "sidebands of sidebands", were discussed in relation to FINESSE. Some cases where the lack of "SBs of SBs" can become troublesome were demonstrated. What to watch out, how to workaround etc. were also discussed. Though the discussions presented here are not the complete ones, I believe they are still good enough to give the readers the basic ideas.

Various things were mentioned, but the most important thing is probably to become critical (while trusting the software; If we don't believe anything then there's no point in using softwares). That's seemingly difficult task, but as far as FINESSE is concerned it's not that bad, partly because we know we can trust it in most of the cases, and partly because the internal of FINESSE is very well documented.

I'd like to give big thanks to Andreas for developing this great software with a nice documentation, and to Gerhard for developing LISO, the basis of FINESSE as well as a very useful software on its own.

6 References

1. FINESSE packages including the documentation are found in Andreas Freise's homepage: `<http://www.rzg.mpg.de/~adf/>`
2. LISO is now in GEO Logbook: `<https://info.geo600.uni-hannover.de/cgi-bin/dcnote.pl?nb=notebook&action=view&page=35>`
3. See FINESSE manual 4.4.2; "Amplitude Modulation".

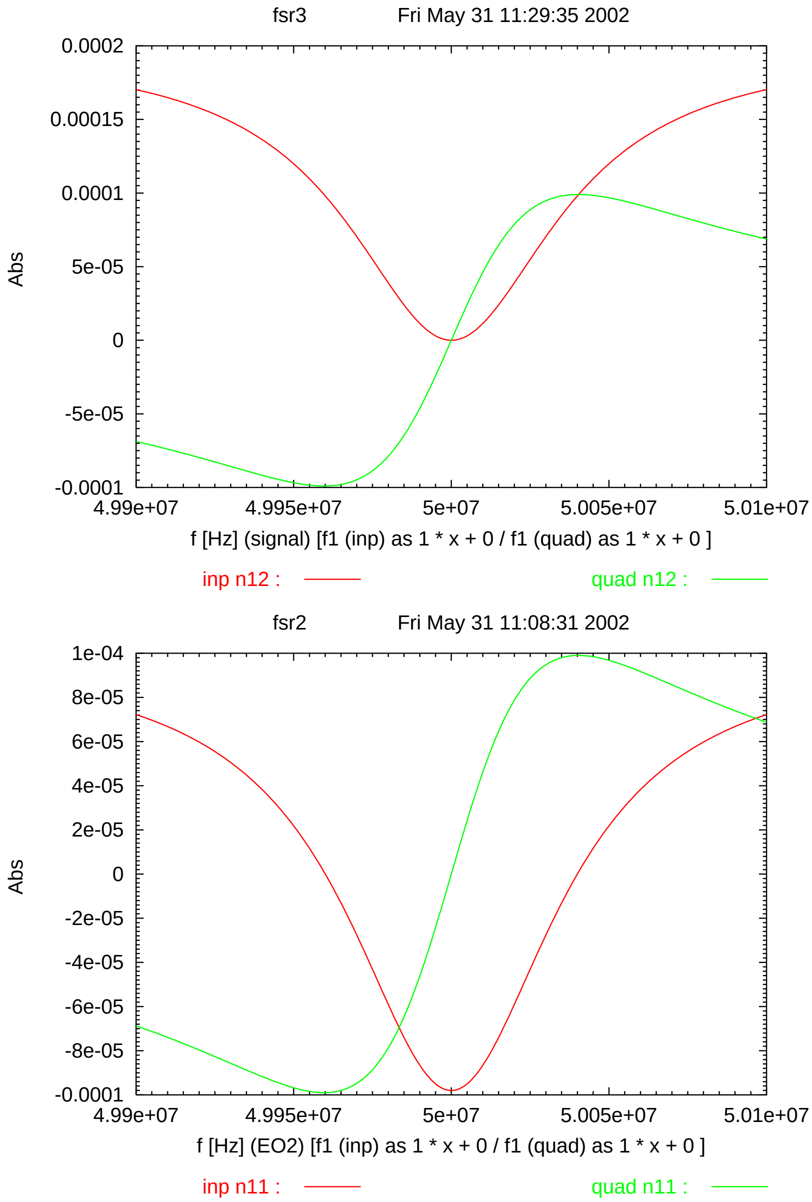

Figure 12: In- and quadrature-phase error signal of a slightly modified Telada's technique. In a real world, there's no signal when the probe frequency is equal to the FSR of the cavity (in this case 50MHz), which is the case with the first (upper) graph that was obtained by using BS instead of EOM to generate probe sidebands. On the other hand, the second (lower) graph was obtained by using an EOM for probe sidebands. Since the lack of "sidebands of sidebands" adds a large "DC error" in the in-phase signal, there's no null-signal point any more.

4. A. Araya, S. Telada, K. Tochikubo, S. Taniguchi, R. Takahashi, K. Kawabe, D. Tatsumi, T. Yamazaki, S. Kawamura, S. Miyoki, S. Moriwaki, M. Musha, S. Nagano, M.-K. Fujimoto, K. Horikoshi, N. Mio, Y. Naito, A. Takamori and K. Yamamoto: "Absolute-Length Determination of a Long-Baseline Fabry-Perot Cavity by Means of Resonating Modulation Sidebands", Appl. Opt. 38 (1999) 2848-2856 .